A New Fixed Point Theorem in Dislocated Quasi b-Metric Spaces

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Abstract: In this manuscript, a new fixed point results involving a single continuous self-map satisfying certain contractive conditions in the setting of dislocated quasi b metric spaces has been established and it’s existence and uniqueness has been established. The present result improves and generalizes several well-known comparable results in the existing literature.

Keywords: Complete Dislocated Quasi –b- Metric Space, Contraction mapping, fixed point, coachy sequence.

1. Introduction

One of the most dynamic research topics in non-linear analysis is fixed point theory. The most important result of this theory is contraction mapping which was proved by the Polish Mathematician Banach (1922) called the Banach contraction mapping principle. This principle has been generalized by various authors by putting different type of contractive conditions either on the mappings or on the spaces Sharma (2018). The celebrated Banach contraction principle was introduced and generalized by (P.Hitzler & Seda, 2000; Hitzler, 2001) in a complete dislocated metric space. Since then, Zeyada et al. (2005) introduced the notion of dislocated quasi metric space for the first time. The most interesting property of this space was that self-distance need not to be necessarily zero. Afterwards, the idea of dislocated quasi b-metric space was presented by Rahman & Sarwar (2016). Recently, Kidane and Aynalem (2019) proved some fixed point result in the setting of dislocated quasi-b-metric spaces.

Inspired by the result of Kidane and Aynalem, the aim of this research work was to establish a new fixed point theorem for maps in the setting of dislocated quasi b-metric spaces which extends, improves and generalizes comparable results in the existing literature. Moreover, we provided an example to support our main result.
2. Preliminaries

Note: Throughout this research work $R^+$ represents the set of non-negative real numbers and $N$ represents the set of natural numbers.

First we remember some known definition and lemma.

**Definition 2.1.** (Rahman & Sarwar, 2016) let $X$ be a non-empty set. A function $d : X \times X \rightarrow [0, \infty)$ said to be dislocated quasi b-metric if the following conditions are satisfied:

1. $d(x; y) = d(y; x) = 0 \Rightarrow x = y$;
2. There exists $k \geq 1$ such that $d(x, y) \leq k[d(x, z) + d(z, y)]$

$\forall x, y, z \in X$ in this case, the pair $(X, d)$ is called a dislocated quasi-b-metric space or in short (dq b) metric spaces.

**Definition 2.2.** (Rahman & Sarwar, 2016) A sequence $\{x_n\}$ in a dislocated quasi b metric space $(X, d)$ is said to converge to a point $x \in X$ if and only if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.$$ 

**Definition 2.3.** (Rahman & Sarwar, 2016) Let $(X, d)$ be a dq b-metric space. Then a sequence $\{x_n\}$ is said be a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0(\varepsilon)$ that is, $\lim_{n \to \infty}(x_n, x_m) = 0$.

**Definition 2.4.** (Rahman & Sarwar, 2016) A dislocated quasi b-metric space $(X, d)$ is called complete if every Cauchy sequence $\{x_n\}$ in $(X, d)$ converges to a point $x \in X$.

**Definition 2.5.** (Rahman & Sarwar, 2016) Let $(X, d_1)$ and $(Y, d_2)$ be two dislocated Quasi b-metric spaces, then the mapping $T : X \rightarrow Y$ is said to be continuous if for each sequence $\{x_n\}$ which is convergent to $x_0$ in $X$, the sequence $Tx_n$ converges to $Tx_0$ in $Y$.

**Lemma 2.1.** (Rahman & Sarwar, 2016) Limit of a convergent sequence in dislocated quasi b-metric space is unique.

**Definition 2.6.** Let $T : X \rightarrow X$ be a self-map in a complete metric space $(X, d)$. Then $T$ is said to be a contraction mapping if there exist a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$

$\forall x, y, z \in X$

**Definition 2.7.** Let $X$ be a nonempty set and $T : X \rightarrow X$ a self-map. We say that $x$ is a fixed point of $T$ if $Tx = x$. 

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**Theorem 2.1.** (Kidane & Aynalem, 2019) Let \((X,d)\) be a complete dislocated quasi b metric space with parameter \(k \geq 1\) and let \(T : X \rightarrow X\) be a continuous self-mapping satisfying the following condition.

\[
d(Tx,Ty) \leq a_1 d(x,y) + a_2 \left[ \frac{d(x,Tx)d(y,Ty)}{d(x,y)} \right] + a_3 [d(x,Tx) + d(y,Ty)] + a_4 [d(x,Ty) + d(y,Tx)] + a_5 [d(x,Tx) + d(x,y)] + a_6 [d(y,Ty) + d(x,y)] + a_7 [d(x,Ty) + d(x,y)] + a_8 [d(y,Tx) + d(x,y)]
\]

\(\forall x, y \in X\) with \(d(x,y) \neq 0\) where \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \geq 0\) and

\[
0 \leq a_1 + 2a_2 + 4a_4 + 2a_5 + 2a_6 + 3a_7 + 3a_8 < 1.
\]

Then \(T\) has a unique fixed point.

### 3. Main Results

**Theorem 3.1.** Let \((X, d)\) be a complete dislocated quasi b metric space with parameter \(k \geq 1\) and let \(T : X \rightarrow X\) be a continuous self-mapping satisfying the following condition.

\[
d(Tx,Ty) \leq a_1 d(x,y) + a_2 \left[ \frac{d(x,Tx)d(y,Ty)}{d(x,y)} \right] + a_3 [d(x,Tx) + d(y,Ty)] + a_4 [d(x,Ty) + d(y,Tx)] + a_5 [d(x,Tx) + d(x,y)] + a_6 [d(y,Ty) + d(x,y)] + a_7 [d(x,Ty) + d(x,y)] + a_8 [d(y,Tx) + d(x,y)]
\]

\(\forall x, y \in X\) with \(d(x,y) \neq 0\) where \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \geq 0\) are non-negative constants with

\[
a_2 + k(a_3 + 2a_5 + a_6) + (k + 1)a_3 + (2k^2 + 2k)a_4 + (k^2 + 2k)(a_7 + a_8) + (k^2 + k + 1)a_9 < 1.
\]

Then \(T\) has a unique fixed point.

**Proof:** Let \(x_0\) be any arbitrary point of \(X\). We can define a sequences \(\{x_n\} \in X\) such that \(x_1 = Tx_0\), \(x_2 = Tx_1, \ldots, x_{n+1} = Tx_n\) for \(n = 0, 1, 2, \ldots\) replacing \(x\) by \(x_{n-1}\) and \(y\) by \(x_n\) in (1) of Theorem 3.1,
we have

\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a_1 d(x_{n-1}, x_n) + a_2 \left[ \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} \right] \\
+ a_3 [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + a_4 [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\
+ a_5 [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)] + a_6 [d(x_n, Tx_n) + d(x_{n-1}, x_n)] \\
+ a_7 [d(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)] + a_8 [d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n)] \\
+ a_9 [d(x_{n-1}, Tx_n) + d(x_n, Tx_n)]. \]

Simplifying the above inequality, we have

\[ d(x_n, x_{n+1}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
+ a_4 [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + a_5 [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)] \\
+ a_6 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + a_7 [d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)] \\
+ a_8 [d(x_n, x_n) + d(x_{n-1}, x_n)] + a_9 [d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})]. \]

Using triangular inequality, we have

\[ d(x_n, x_{n+1}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
+ a_4 k[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
+ a_5 [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)] + a_6 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
+ a_7 [k(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + d(x_{n-1}, x_n)] \\
+ a_8 [k(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + d(x_{n-1}, x_n)] \\
+ a_9 [k(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + d(x_{n-1}, x_n)]. \]
Simplification yields

\[ d(x_n, x_{n+1}) \leq \frac{a_1 + a_3 + 2(k a_4 + a_5) + a_6 + (a_7 + a_8)(k+1) + k a_9}{1 - (a_2 + a_3 + a_6 + k(2a_4 + a_7 + a_8) + (k+1)a_9)} d(x_{n-1}, x_n). \]

\[ d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n). \]

Where \( \lambda = \frac{a_1 + a_3 + 2(k a_4 + a_5) + a_6 + (a_7 + a_8)(k+1) + k a_9}{1 - (a_2 + a_3 + a_6 + k(2a_4 + a_7 + a_8) + (k+1)a_9)} < \frac{1}{k}. \)

Also, we can show that \( d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1}). \)

Proceeding in similar fashion, we get

\[ d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \]

Now, to show that \( \{x_n\} \) is a Cauchy sequence consider for any two non-negative integers \( m \) and \( n \) with \( m > n \) and using \( (d_2) \) we have,

\[ d(x_n, x_m) \leq k d(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + \cdots + k^{m-n} d(x_{m-1}, x_m). \]

\[ \leq k\lambda^n d(x_0, x_1) + k^{2}\lambda^{n+1} d(x_0, x_1) + k^{m-n}\lambda^{m-1} d(x_0, x_1). \]

\[ = k\lambda^n (1 + k\lambda + (k\lambda)^2 + \cdots + (k\lambda)^{m-n-1}) d(x_0, x_1). \]

\[ \leq \frac{k\lambda^n}{1-k\lambda} d(x_0, x_1). \]

\[ = \frac{k\lambda^n}{1-k\lambda} \to 0 \text{ as } n \to \infty. \]

which implies that,

\[ d(x_n, x_m) \to 0 \text{ as } n, m \to \infty. \]

This shows that \( \{x_n\} \) is a Cauchy sequence in a complete dislocated quasi b metric space.

As a result there must exist a point \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \).

Furthermore, since \( T \) is continuous, thus we have

\[ T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = Tx^*. \]

\[ \lim_{n \to \infty} x_{n+1} = Tx^*. \]

This implies that

\[ Tx^* = \lim_{n \to \infty} x_{n+1}. \]

Thus \( Tx^* = x^* \). Therefore, \( x^* \) is the fixed point of \( T \).
**Uniqueness:** Now, we show that $x^*$ is a unique fixed point of $T$.

Suppose there is another fixed point $y^*$ of $T$ such that $Ty^* = y^*$ for $x^* \neq y^*$. Consider

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

If $x^*$ is fixed point of $T$ then $d(x^*, x^*) = 0$ by using (1). Similarly, $d(y^*, y^*) = 0$.

So, we have the following inequality

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\leq a_1 d(x^*, y^*) + a_2 \left[ \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{d(x^*, y^*)} \right] + a_3 [d(x^*, Tx^*) + d(y^*, Ty^*)]$$

$$+ a_4 [d(x^*, Ty^*) + d(y^*, Tx^*)] + a_5 [d(x^*, Tx^*) + d(x^*, y^*)] + a_6 [d(y^*, Ty^*) + d(x^*, y^*)]$$

$$+ a_7 [d(x^*, Ty^*) + d(y^*, Tx^*)] + a_9 [d(x^*, Ty^*) + d(y^*, y^*)].$$

$$= a_1 d(x^*, y^*) + a_2 \left[ \frac{d(x^*, x^*)d(y^*, y^*)}{d(x^*, y^*)} \right] + a_3 [d(x^*, x^*) + d(y^*, y^*)]$$

$$+ a_4 [d(x^*, y^*) + d(y^*, x^*)] + a_5 [d(x^*, x^*) + d(x^*, y^*)] + a_6 [d(y^*, y^*) + d(x^*, y^*)]$$

$$+ a_7 [d(x^*, y^*) + d(x^*, y^*)] + a_9 [d(x^*, y^*) + d(y^*, y^*)].$$

As $x^*$ and $y^*$ are fixed points of $T$ from the above inequality, we have

$$d(x^*, x^*) = 0$$

$$d(y^*, y^*) = 0.$$

Hence, we get

$$d(x^*, y^*) = a_1 d(x^*, y^*) + a_4 [d(x^*, y^*) + d(y^*, x^*)] + a_5 d(x^*, y^*)$$

$$+ a_6 d(x^*, y^*) + a_7 d(x^*, y^*) + a_9 d(x^*, y^*).$$

$$d(x^*, y^*) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8 + a_9) d(x^*, y^*) + (a_4 + a_6) d(y^*, x^*). \quad (3.2)$$

Similarly, we have

$$d(y^*, x^*) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8 + a_9) d(y^*, x^*) + (a_4 + a_6) d(x^*, y^*). \quad (3.3)$$

Adding equation (3.2) and (3.3), we have

$$[d(x^*, y^*) + d(y^*, x^*]) \leq (a_1 + 2(a_4 + a_6) + a_5 + a_6 + 2a_7 + a_9) [d(x^*, y^*) + d(y^*, x^*)]. \quad (3.4)$$
Since, \(a_1 + 2(a_4 + a_9) + a_5 + a_6 + 2a_7 + a_9\) \(\leq 1\), hence inequality (3.4) is possible only if 
\[d(x^*, y^*) + d(y^*, x^*) = 0.\]

this implies that 
\[d(x^*, y^*) = d(y^*, x^*)\]

Implies 
\[d(x^*, y^*) = 0.\]

Similarly, 
\[d(y^*, x^*) = 0 \Rightarrow x^* = y^*.\]

Therefore \(T\) has a unique fixed point.

**Remark 3.1** If we put \(k = 1\) and \(a_9 = 0\), then we get Theorem 2.1.

### 4. Conclusion

In 2019 Kidane and Aynalem have been established and proved the existence and uniqueness of fixed point and common fixed point results for certain maps in dislocated quasi metric (dq-metric) spaces.

In this research work, we established and proved the existence and uniqueness of a new fixed point result for maps in the perspective of complete dislocated quasi \(b\)-metric spaces.

The derived result generalizes and extends several well-known comparable results in the existing literature. Also, we provided an example in support of the main result.

**References:**